Relativistic Electrodynamics

M.Sc. 2nd Semester

MPHYCC-6: Electrodynamics and Plasma Physics

Unit IV (Part 1)

Dr. Ashok Kumar Jha

Assistant Professor

Department of Physics, Patna University

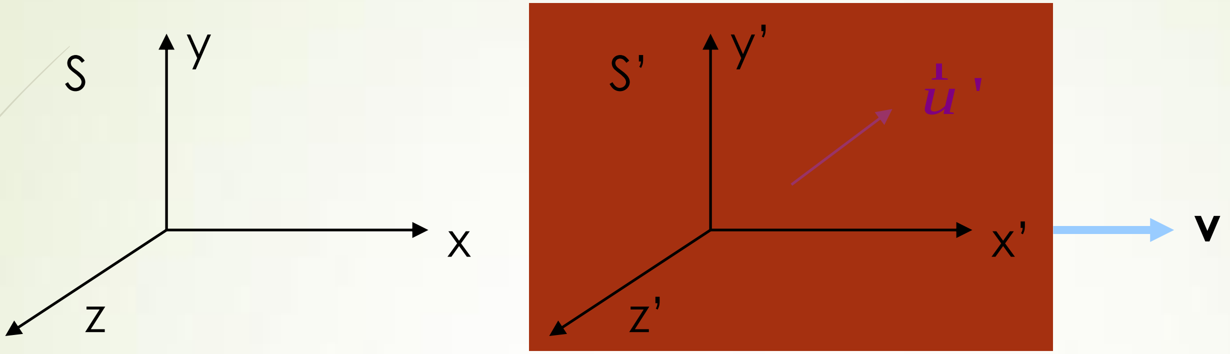
Mob:7903067108, Email: ashok.jha1984@gmail.com

Lecture Outline

- Special Theory of Relativity
- Transformation of electric force
- 4-vector notations
- Covariant 4-vector and contravariant 4-vector
- Covariant form of the electromagnetic field
- Covariant form of the Maxwell equations

- Assignment

Special theory of relativity



energy-momentum

$$p = (p_0, \vec{p}) = (E/c, \vec{p})$$

$$E = \gamma_u mc^2, \ \vec{p} = \gamma_u m\vec{u}$$

space-time

$$x = (x_0, \vec{x}) = (ct, \vec{x})$$

$$x_0 = \gamma(x'_0 + \beta x'_1)$$
$$x_1 = \gamma(x'_1 + \beta x'_0)$$
$$x_2 = x'_2$$
$$x_3 = x'_3$$

$$p_0 = \gamma_v(p'_0 + \beta p'_1)$$
$$p_1 = \gamma_v(p'_1 + \beta p'_0)$$
$$p_2 = p'_2$$
$$p_3 = p'_3$$

$$\beta \equiv v/c, \quad \gamma \equiv \frac{1}{\sqrt{1-(v/c)^2}}$$

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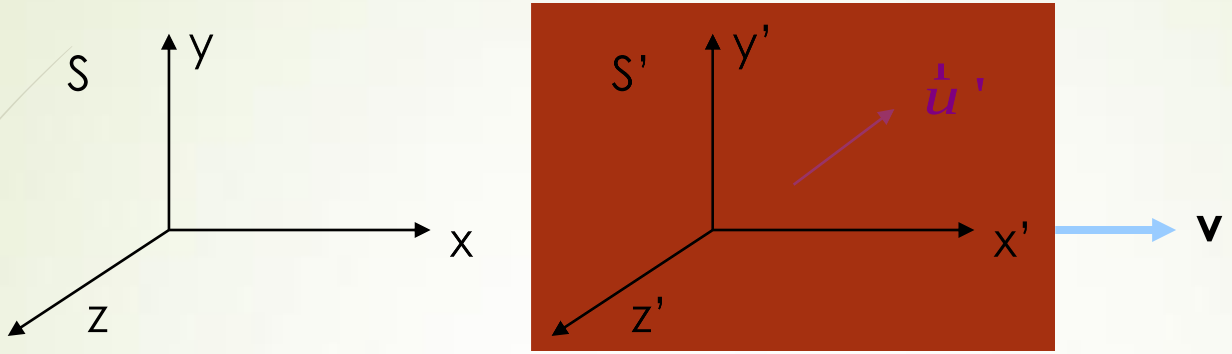


velocity, $\vec{u} = d\vec{x} / dt$

$$u_1 = \frac{u'_1 + v}{1 + \frac{u'_1 v}{c^2}}$$

$$u_2 = \frac{u'_2}{\gamma\left(1 + \frac{u'_1 v}{c^2}\right)}$$

$$u_3 = \frac{u'_3}{\gamma\left(1 + \frac{u'_1 v}{c^2}\right)}$$

force, $\vec{F} = d\vec{p} / dt$

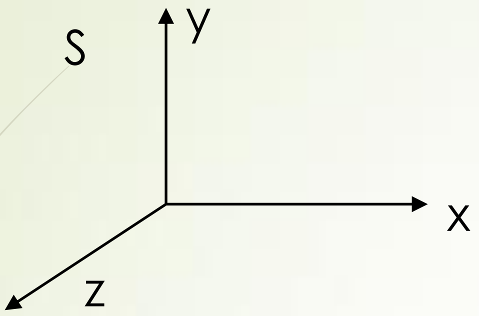
$$F_1 = F'_1 + \frac{v}{c^2 + u'_1 v}\left(u'_2 F'_2 + u'_3 F'_3\right)$$

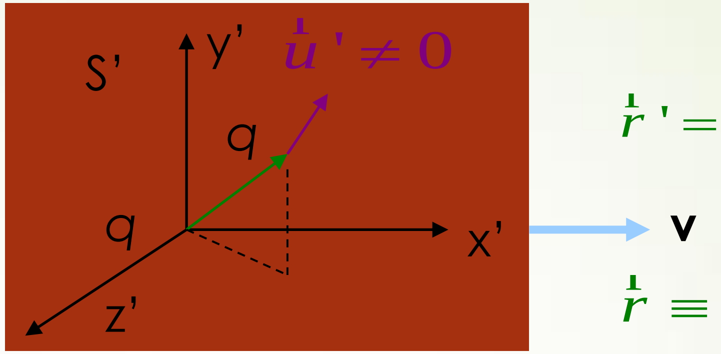
$$F_2 = \frac{F'_2}{\gamma\left(1 + \frac{u'_1 v}{c^2}\right)}$$

$$F_3 = \frac{F'_3}{\gamma\left(1 + \frac{u'_1 v}{c^2}\right)}$$

$$\left(\frac{dE'}{dt'} = \vec{F}' \cdot \vec{u}' \text{ has been used}\right)$$

Transformation of electric force





$$\vec{r}\,' = \gamma(x - vt)\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{r} \equiv (x - vt)\hat{x} + y\hat{y} + z\hat{z}$$

$$F'_x = \frac{q^2 x'}{r'^3},\ F'_y = \frac{q^2 y'}{r'^3},\ F'_z = \frac{q^2 z'}{r'^3}$$

$$\Rightarrow F_x = F'_x + \frac{v}{c^2 + u'_x v}\left(u'_y F'_y + u'_z F'_z\right)$$

$$\text{K} \quad = \frac{e^2\gamma}{r'^3}\left(x - vt + \frac{u_y v}{c^2} y + \frac{u_z v}{c^2} z\right)$$

$$F_y = \frac{F'_y}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)} = \frac{e^2 \gamma y}{r'^3}\left(1 - \frac{u_x v}{c^2}\right)$$

$$F_z = \frac{F'_z}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)} = \frac{e^2 \gamma z}{r'^3}\left(1 - \frac{u_x v}{c^2}\right)$$

Vector notation

$$\vec{F} = \frac{q^2\gamma}{r'^3}\left\{\vec{r} + \frac{v}{c^2}\left[\left(u_y y + u_z z\right)\hat{x} - u_x\left(y\hat{y} + z\hat{z}\right)\right]\right\}$$

$$= \frac{q^2\gamma}{r'^3}\left\{\vec{r} + \frac{1}{c^2}\vec{u} \times (\vec{v} \times \vec{r})\right\}$$

$$= q\left(\vec{E} + \frac{\vec{u}}{c} \times \vec{B}\right)$$

where $\vec{E} = \frac{q\gamma}{r'^3}\vec{r}$

$$\vec{B} \equiv \frac{q\gamma}{r'^3}\left(\frac{\vec{v}}{c} \times \vec{r}\right) = \frac{\vec{v}}{c} \times \vec{E}$$

($\vec{u} = \vec{v}$ on previous page.)

4-vector notations

- Space-time (ct, **x**)
- energy-momentum (E/c, ***p***)
- scalar potential-vector potential (φ, ***A***)
- 4-dim space-time operator (below)
- charge density-charge current (below)
- …

but not the usual velocity, acceleration, force…

4-dim space-time operator

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_0}, \frac{\partial}{\partial \vec{x}} \right) \text{ or } \left(\partial_0, \partial_{\vec{x}} \right)$$

$$\partial_0 = \gamma\left(\partial'_0 - \beta \partial'_1\right)$$
$$\partial_1 = \gamma\left(\partial'_1 - \beta \partial'_0\right)$$
$$\partial_2 = \partial'_2$$
$$\partial_3 = \partial'_3$$

charge density-charge current

$$j = (j_0, \vec{j}) = (c\rho, \vec{j})$$
$$dq = \rho dV, \ \vec{j} = \rho \vec{u}$$

$$j_0 = \gamma_v (j'_0 + \beta j'_1)$$
$$j_1 = \gamma_v (j'_1 + \beta j'_0)$$
$$j_2 = j'_2$$
$$j_3 = j'_3$$

Covariant 4-vector and contravariant 4-vector

Vectors that transform like x are called contravariant vectors; Vectors that transform like $\partial / \partial x$ are called covariant vectors. Their notations are distinguished by the position of the index.

$$dx^{\alpha} = \frac{\partial x^{\alpha}}{\partial x'^{\beta}} dx'^{\beta}$$

$$\frac{\partial}{\partial x^{\alpha}} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \frac{\partial}{\partial x'^{\beta}} \quad \text{or} \quad \partial_{\alpha} = \frac{\partial x'^{\beta}}{\partial x^{\alpha}} \partial'_{\beta}$$

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \xleftrightarrow{\text{inverse}} \quad \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Raising and lowering of the index

$$A_{\alpha} = g_{\alpha\beta} A^{\beta}$$

$$A^{\alpha} = g^{\alpha\beta} A_{\beta}$$

$$\text{where } g_{\alpha\beta} = g^{\alpha\beta} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

g (called a metric tensor) converts a contravariant vector to a covariant vector, and vice versa

Inner product between two 4-vectors

$$A^\alpha = (A^0, A^i),\ B^\alpha = (B^0, B^i)$$

$$A \cdot B \equiv A^\alpha B_\alpha = A_\alpha B^\beta = A^\alpha g_{\alpha\beta} B^\beta = A_\alpha g^{\alpha\beta} B_\beta$$

$$= A^0 B_0 + A^i B_i$$

$$= A^0 B^0 - A^i B^i$$

The inner product is invariant under Lorentz transformation because A^α and transform oppositely. B_α

Relativistic invariants

$$x^2 = c^2 t^2 - |\vec{x}|^2$$

$$p^2 = (E/c)^2 - |\vec{p}|^2 = m^2 c^2$$

$$\partial_\alpha j^\alpha = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

… etc

Conversely, any linear transformation that leaves x^2 (or other inner product) invariant must be a Lorentz transformation (including spatial rotation).

Analogy: any linear transformation that leaves $|\mathbf{x}|^2$ (or other inner product) invariant must be a rotation.

Covariant form of the electromagnetic field

$$\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\phi; \qquad \vec{B} = \nabla \times \vec{A}$$

can be written as

$$E_i = -(\partial^0 A^i - \partial^i A^0); \quad \varepsilon^{ijk} B_k = -(\partial^i A^j - \partial^j A^i)$$

it's natural to define

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$B_k = -\frac{1}{2}\varepsilon_{ijk} F^{ij}$$

Transformation of the field strength tensor

$$F^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^\gamma}\frac{\partial x^\beta}{\partial x'^\delta} F'^{\gamma\delta}$$

or

$$\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Covariant form of the Maxwell equations

$$\nabla\cdot\vec{E}=4\pi\rho$$
$$\nabla\cdot\vec{B}=0$$
$$\nabla\times\vec{E}+\frac{1}{c}\frac{\partial\vec{B}}{\partial t}=0$$
$$\nabla\times\vec{B}-\frac{1}{c}\frac{\partial\vec{E}}{\partial t}=\frac{4\pi}{c}\vec{J}$$

$\Longleftrightarrow$

$$\partial_\alpha F^{\alpha\beta}=\frac{4\pi}{c}J^\beta \quad \Rightarrow \partial_\alpha J^\alpha=0$$
$$\partial^\alpha F^{\beta\gamma}+\partial^\beta F^{\gamma\alpha}+\partial^\gamma F^{\alpha\beta}=0$$
(Bianchi identity)

Covariant form of the Lorentz force equation

The usual velocity v is not part of a 4-vector since *t* is not invariant under the Lorentz transformation

4-velocity $u^\alpha\equiv\dfrac{dx^\alpha}{d\tau}$

where $d\tau$ is a relativistic inv. w/ the dimension of time

choose $d\tau=\dfrac{ds}{c}\quad\left(ds^2=c^2dt^2-|d\vec{x}|^2\right)$

$$=dt\sqrt{1-\beta^2}=\frac{dt}{\gamma}\quad(=dt' \text{ when } \vec{x}'=0)$$

$$u=(\gamma c,\gamma\vec{v})$$

$$\frac{d\vec{p}}{dt}=q\left(\vec{E}+\frac{\vec{v}}{c}\times\vec{B}\right)$$
$$\Leftrightarrow\frac{dp^i}{d\tau}=\frac{q}{c}F^{i\beta}u_\beta$$
$$\frac{dp^0}{d\tau}=\frac{q}{c}F^{0\beta}u_\beta$$
$$\Leftrightarrow\frac{dE}{dt}=q\vec{E}\cdot\vec{v}$$

Relativistic electrodynamics

$$\partial_\alpha F^{\alpha\beta} = \frac{4\pi}{c} J^\beta$$

$$\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$$

$$\frac{dp^\alpha}{d\tau} = \frac{q}{c} F^{\alpha\beta} u_\beta$$

Exactly the same as Maxwell's electrodynamics!

Assignment

1. Describe the transformation of force according to special theory of relativity.
2. Express Maxwell's equations in covariant form.
3. Describe the meaning of 4-vector notation.
4. Express Electromagnetic field in covariant form.

Thank You